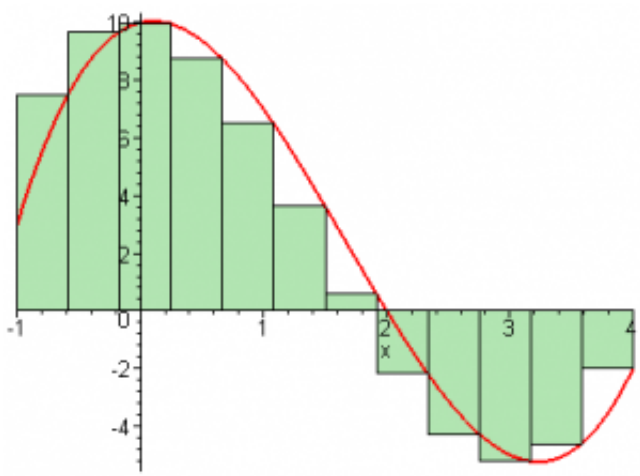
Sections 5.2 & 5.3

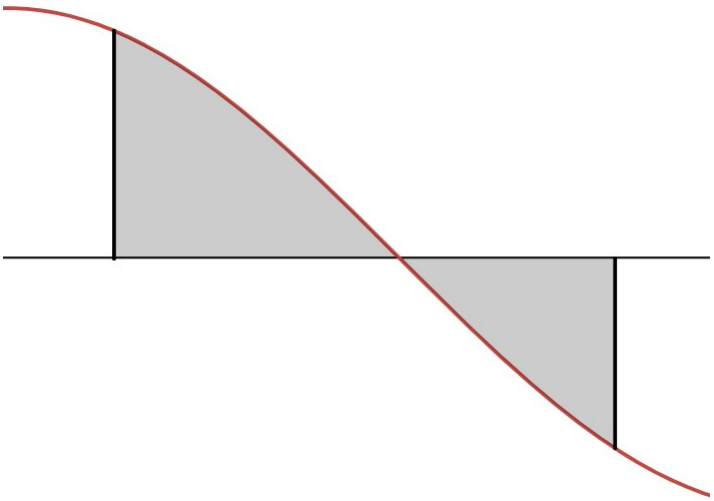
The Definite Integral

We saw in Section 5.1 that a Riemann sum of the function f over an interval $[a, b]$ approximates the net change of any one of the antiderivatives of f. When investigating the graph of f along with a Riemann sum it appeared as though the Riemann sum is also approximating the area of a region bounded by f.



When $f(x_i^*) < 0$, the corresponding term of the Riemann sum is negative. The notion of area we are familiar with from geometry does not allow negative areas. So, what geometric measure is the Riemann sum computing?

Definition 5.2.1. Consider the region R bounded by the graph of a continuous function f and the x-axis from $x = a$ to $x = b$. The *signed area* (or net area) of R is the sum of the areas of the parts of R above the x-axis minus the sum of the areas of the parts of R below the x-axis.



We want the net area to be expressed by a limit of Riemann sums as the number of subintervals goes to infinity.

Definition 5.2.2. Suppose $[x_0, x_1], [x_1, x_2], [x_2, x_3], dots, [x_{n-1}, x_n]$ are subintervals of $[a, b]$ with

$$a = x_0 < x_i < x_{i+1} < x_n = b.$$

Let $\Delta x_k = x_k - x_{k-1}$ and let x_k^* be a point in $[x_{k-1}, x_k]$. If f is defined on $[a, b]$, then the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x_k = f(x_1^*)\Delta x_1 + f(x_2^*)\Delta x_2 + \cdots + f(x_n^*)\Delta x_n$$

is the *general Riemann sum* of f on $[a, b]$.

The general Riemann sum is what we need to ensure the value we get from limiting a particular sum will actually give the signed area.

Definition 5.2.3. Let $\Delta = \max\{\Delta x_1, \Delta x_2, \ldots \Delta x_n\}$. If f is defined on $[a, b]$ we say f is *integrable* on $[a, b]$ if $\lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$ exists and is unique over all partitions of $[a, b]$ and all choices of sample points x_k^*. This limit is the *definite integral* of f from a to b denoted

$$\text{signed area} = \int_a^b f(x)\, dx = \lim_{\Delta \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

Theorem 5.2.4 (Integrable Functions)**.** If f is continuous on $[a, b]$ or bounded on $[a, b]$ with a finite number of discontinuities, the f is integrable on $[a, b]$.

What Theorem 5.2.4 means for us: we can choose one of the left, right, midpoint sums to compute the limit giving the definite integral!

Example 5.2.5. Find the signed area under $f(x) = 3x + 5$ from $x = 1$ to $x = 2$.

Example 5.2.6. Compute $\int_0^{2\pi} \sin x \, dx$

In general, limits of Riemann sums are not an easy method to compute definite integrals. We have already seen the Riemann sums approximate net change of derivatives:

Theorem 5.2.7 (The Fundamental Theorem of Calculus, Part II)**.** If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$ then

$$\int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

Example 5.2.8. Compute the indicated definite integrals.

1. $\displaystyle\int_1^2 (3x + 5)dx$

2. $\displaystyle\int_0^2 (x^3 + 1)dx$

3. $\int_0^{2\pi} \cos\theta \, d\theta$

4. $\int_{-4}^{4} \sqrt{16 - t^2} \, dt$

A few things are interesting with part 4) above. First, we don't know what the antiderivative of $\sqrt{16-x^2}$ is, but we were still able to compute the definite integral (a.k.a. the signed area under the curve). We can construct an antiderivative of a function via the definite integral:

Definition 5.2.9. Let f be a continuous function for $t \geq a$. The *area function* for f with left endpoint a is

$$A(x) = \int_a^x f(t)\,dt.$$

The area function gives the net area of the region bounded by f and the x-axis from a to x.

Theorem 5.2.10 (The Fundamental Theorem of Calculus, Part I)**.** If f is continuous on $[a, b]$, then the area function

$$A(x) = \int_a^x f(t)\,dt$$

is continuous on $[a, b]$ and differentiable on (a, b). The area function satisfies $A'(x) = f(x)$. That is,

$$A'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x).$$

Example 5.2.11. Compute the derivative:

1. $\dfrac{d}{dx}\displaystyle\int_0^x \left(3t^2 + \sqrt{t}\right)dt$

2. $\dfrac{d}{dx}\displaystyle\int_{-1}^{x} e^{-t^2}\,dt$

3. $\dfrac{d}{dx}\displaystyle\int_{1}^{3} (3t+5)dt$

4. $\dfrac{d}{dx}\displaystyle\int_{0}^{x^2} \sin t\,dt$

Integral Properties

Returning to integrating $\sqrt{16-t^2}$, if all we want is to find the area and do not know an antiderivative, then we want to leverage as much geometry/function properties/integral properties as possible.

Theorem 5.2.12 (Integral Properties)**.**

1. $\displaystyle\int_a^a f(x)\,dx = 0$
2. $\displaystyle\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ (Reversing limits)
3. $\displaystyle\int_a^b \big(f(x) \pm g(x)\big)dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ (Sum Rule)
4. $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ (Constant Multiple Rule)
5. $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ (Adjacent Region Rule)
6. $\displaystyle\int_a^b |f(x)|dx$ is the geometric area of the region bounded by f and the x-axis.

Theorem 5.2.13 (Function Properties)**.**

- If f is odd, then $\displaystyle\int_0^a f(x)\,dx = -\int_{-a}^0 f(x)\,dx.$
- If f is even, then $\displaystyle\int_0^a f(x)\,dx = \int_{-a}^0 f(x)\,dx.$
- If f has period k, then $\displaystyle\int_x^{x+nk} f(x)\,dx = n\int_x^{x+k} f(x)\,dx.$

Example 5.2.14.

1. $\displaystyle\int_{-100}^{100} x^5\,dx$

2. $\int_{\pi}^{\pi} x \cos x \, dx$

3. $\int_{0}^{2\pi} |\sin x| dx$

4. If $I = \int_{0}^{1} \big(x^3 - 2x\big) dx = -\frac{3}{4}$, find

   (a) $\int_{0}^{1} \big(4x - 2x^3\big) dx$

   (b) $\int_{1}^{0} \big(2x - x^3\big) dx$

5. If f and g are defined on $[1, 6]$ such that $\int_1^6 f(x)\,dx = 10$, $\int_1^6 g(x)\,dx = 5$, $\int_4^6 f(x)\,dx = 5$, and $\int_1^4 g(x)\,dx = 2$, find

(a) $\int_1^4 f(x)\,dx$

(b) $\int_1^6 \big(f(x) - g(x)\big)dx$

(c) $\int_4^1 2f(x)dx$

(d) $\int_4^6 \big(g(x) - f(x)\big)dx$

Integral Inequalities

We can use inequalities to compare definite integrals, draw graphs of each to convince yourself of their validity.

Theorem 5.2.15 (Integral Inequalities)**.**

1. If $f(x) \geq 0$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \geq 0$.
2. If $f(x) \geq g(x)$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.
3. If $m \leq f(x) \leq M$ on $[a, b]$, then $m(b-a) \leq \displaystyle\int_a^b f(x)\,dx \leq M(b-a)$.

Example 5.2.16. Show that for all $r > 1$, $\displaystyle\int_0^1 x^r\,dx \leq \frac{1}{2}$.